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Static observables of relativistic three-fermion systems with instantaneous interactions

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Abstract. We show that static properties like the charge radius and the magnetic moment of relativistic three-fermion bound states with instantaneous interactions can be formulated as expectation values with respect to intrinsically defined wave functions. The resulting operators can be given a natural physical interpretation in accordance with relativistic covariance. We also indicate how the formalism may be generalized to arbitrary moments. The method is applied to the computation of static baryon properties with numerical results for the nucleon charge radii and the baryon octet magnetic moments. In addition, we make predictions for the magnetic moments of some selected nucleon resonances and discuss the decomposition of the nucleon magnetic moments in contributions of spin and angular momentum, as well as the evolution of these contributions with decreasing quark mass.

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1 Introduction

Static observables of bound-state systems in field-theoretic descriptions are usually extracted from form factors in the limit of vanishing squared four-momentum transfer of the probing exchange particle. For example, the mean square charge radius is defined as the slope of the electric form factor at $Q^2 = 0$ and the magnetic moment is the value of the magnetic form factor at the photon point. Although such an approach is suitable to produce pure numbers it hardly leads to any insight into the underlying structure of the observable. It is, for example, well known how a fermion produces a magnetic moment through both its spin and its angular motion, but how does that translate into the magnetic moment of a bound state, *e.g.*, a baryon composed of three quarks?

On the other hand, static properties in non-relativistic quantum mechanics can be formulated by means of expectation values involving essentially scalar products of wave functions. The non-relativistic charge radius of a composite system of N particles for example is given by

$$\langle r^2 \rangle = \frac{\langle \psi | \sum_{i=1}^{N} q_i (\boldsymbol{x}_i - \boldsymbol{R})^2 | \psi \rangle}{Q \langle \psi | \psi \rangle}, \tag{1}$$

where q_i is the charge of particle i, $\boldsymbol{x}_i$ its position, $\boldsymbol{R}$ the center-of-mass coordinate and Q the net charge of the system. A direct relativistic generalization of this expression is unknown. In this paper we will focus our attention to a quark model description of baryons. The generalization to other systems is then quite obvious. We will show that a synthesis of both approaches mentioned before is indeed possible —at least if certain restrictions are made to the kind of interactions between the constituents of the bound system— and leads to new insights into the structure of static properties. Moreover, the actual computation of static moments is then easier and also numerically more reliable in comparison to the computation of form factors in the limit $Q^2 \to 0$.

We work within the framework of the Bethe-Salpeter equation which has been successfully applied to, *e.g.*, baryon mass spectra [1–3] and form factors [4,5].

First, we briefly outline the Bethe-Salpeter formalism. Details may be found in ref. [1]. The construction of current matrix elements from Bethe-Salpeter amplitudes is addressed, of which a detailed discussion is given in [4]. Since by far the most interesting static observables for physical applications are mean square charge radii and magnetic moments, we show how they can be formulated as expectation values with respect to Salpeter amplitudes. These amplitudes turn out to be the natural quantities which replace the non-relativistic wave functions and possess a canonical scalar product. Static observables are then represented by certain well-defined operators whose expectation values are computed with the help of this scalar

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product. The emerging operators can be given a natural physical interpretation and show very interesting structures. To demonstrate the relevance of this formalism, we apply it to a concrete physical model for baryons described in refs. [1–3] to compute nucleon charge radii and magnetic moments and indicate how higher moments can in principle be computed.

2 Bethe-Salpeter equation and current matrix elements

2.1 Bethe-Salpeter equation

The basic quantity which describes three fermions as a bound state is the Bethe-Salpeter amplitude which is defined in position space through:

$$\chi_{\bar{P}a_1a_2a_3} := \langle 0|T\psi^1_{a_1}(x_1)\psi^2_{a_2}(x_2)\psi^3_{a_3}(x_3)|\bar{P}\rangle, \qquad (2)$$

where $\psi^i_{a_i}(x_i)$ are fermion field operators given in the Heisenberg picture and a_i are multi-indices in Dirac space and any internal space which represents a degree of freedom the particle may have. T is the time ordering operator. Here $|0\rangle$ denotes the physical *i.e.* interacting vacuum and $|\bar{P}\rangle$ denotes a three-fermion bound state with total four-momentum $\bar{P}$ on the mass shell, *i.e.* $\bar{P}^2 = M^2$. Because of translational invariance it is convenient to introduce a center-of-mass coordinate X and so-called Jacobi coordinates ξ and η:

$$\begin{aligned} X &:= \tfrac{1}{3}(x_1+x_2+x_3), & x_1 &= X + \tfrac{1}{2}\xi + \tfrac{1}{3}\eta\,,\\ \xi &:= x_1 - x_2, & x_2 &= X - \tfrac{1}{2}\xi + \tfrac{1}{3}\eta\,,\\ \eta &:= \tfrac{1}{2}(x_1+x_2-2x_3), & x_3 &= X - \tfrac{2}{3}\eta\,. \end{aligned} \qquad (3)$$

The corresponding conjugate momenta are then given by the total four-momentum P and the two relative momenta p_ξ and p_η:

$$\begin{aligned} P &:= p_1+p_2+p_3, & p_1 &= \tfrac{1}{3}P + p_\xi + \tfrac{1}{2}p_\eta\,,\\ p_\xi &:= \tfrac{1}{2}(p_1 - p_2), & p_2 &= \tfrac{1}{3}P - p_\xi + \tfrac{1}{2}p_\eta\,,\\ p_\eta &:= \tfrac{1}{3}(p_1+p_2-2p_3), & p_3 &= \tfrac{1}{3}P - p_\eta\,. \end{aligned} \qquad (4)$$

Using this set of coordinates the total momentum dependence factorizes and we may define the Bethe-Salpeter amplitude in momentum space depending only on the relative momenta:

$$\chi_{\bar{P}a_1a_2a_3}(x_1,x_2,x_3) =: e^{-\mathrm{i}\langle\bar{P},X\rangle}\int \frac{\mathrm{d}^4p_\xi}{(2\pi)^4}\frac{\mathrm{d}^4p_\eta}{(2\pi)^4} \times e^{-\mathrm{i}\langle p_\xi,\xi\rangle}e^{-\mathrm{i}\langle p_\eta,\eta\rangle}\chi_{\bar{P}a_1a_2a_3}(p_\xi,p_\eta). \qquad (5)$$

One may write the sum of all two-body interactions in the form of a three-body interaction kernel:

$$\begin{aligned}\bar{K}^{(2)}_{a_1a_2a_3;a_1'a_2'a_3'} := {} & K^{(2)}_{12\;a_1a_2;a_1'a_2'}\big(\tfrac{2}{3}\bar{P}+p_\eta,p_\xi,p_\xi'\big)\\ &\times S^{3\;-1}_{F\,a_3a_3'}(2\pi)^4\delta^{(4)}(p_\eta-p_\eta')\\ &+\text{cycl. perm.}\end{aligned} \qquad (6)$$

Then by introducing the three-particle propagator

$$\begin{aligned} G_{0\,a_1a_2a_3;a_1'a_2'a_3'}(P,p_\xi,p_\eta,p_\xi',p_\eta') &:= (2\pi)^4\delta^{(4)}(p_\xi-p_\xi')\\ &\times(2\pi)^4\delta^{(4)}(p_\eta-p_\eta')S^1_{F\,a_1a_1'}\big(\tfrac{1}{3}\bar{P}+p_\xi+\tfrac{1}{2}p_\eta\big)\\ &\times S^2_{F\,a_2a_2'}\big(\tfrac{1}{3}\bar{P}-p_\xi+\tfrac{1}{2}p_\eta\big)S^3_{F\,a_3a_3'}\big(\tfrac{1}{3}\bar{P}-p_\eta\big), \end{aligned} \qquad (7)$$

the Bethe-Salpeter equation can be written in a compact notation

$$\chi_{\bar{P}} = -\mathrm{i}G_0\left(K^{(3)}+\bar{K}^{(2)}\right)\chi_{\bar{P}}, \qquad (8)$$

where a summation over multi-indices and momentum integrations is tacitly understood.

The Bethe-Salpeter equation would be incomplete without a prescription of how to normalize its solutions. Such a prescription can indeed be found (see ref. [1]). In a covariant form and using our compact notation it reads

$$-\mathrm{i}\overline{\chi}_{\bar{P}}\left[P^\mu\frac{\partial}{\partial P^\mu}\left(G_0^{-1}+\mathrm{i}K^{(3)}+\mathrm{i}K^{(2)}\right)\right]_{P=\bar{P}}\chi_{\bar{P}} = 2M^2, \qquad (9)$$

where G_0^{-1} is the inverse three-particle propagator, *i.e.* the inverse of (7). Here we introduced the adjoint Bethe-Salpeter amplitude defined as

$$\overline{\chi}_{\bar{P}a_1a_2a_3} := \langle\bar{P}|T\bar{\psi}^1_{a_1}(x_1)\bar{\psi}^2_{a_2}(x_2)\bar{\psi}^3_{a_3}(x_3)|0\rangle. \qquad (10)$$

2.2 Salpeter equation

In order to solve the Bethe-Salpeter equation in physical cases relevant for, *e.g.*, the structure of hadrons, one applies two approximations. First, the full propagators are replaced by the free ones:

$$S^i_F(p_i) = \frac{\mathrm{i}}{\not{p}_i - m_i + \mathrm{i}\epsilon}\,. \qquad (11)$$

This approximation accounts for self-energy contributions merely by the introduction of effective fermion masses m_i. Neglecting retardation effects in the interaction kernels leads to the second, so-called, instantaneous approximation. This assumes that there is no dependence of the interaction kernels on the relative energies in the rest frame of the composite system:

$$K^{(3)}(P,p_\xi,p_\eta,p_\xi',p_\eta')\Big|_{P=(M,\mathbf{0})} = V^{(3)}(\boldsymbol{p}_\xi,\boldsymbol{p}_\eta,\boldsymbol{p}_\xi',\boldsymbol{p}_\eta') \qquad (12)$$

$$K^{(2)}\big(\tfrac{2}{3}P+p_\eta,p_\xi,p_\xi'\big)\Big|_{P=(M,\mathbf{0})} = V^{(2)}(\boldsymbol{p}_\xi,\boldsymbol{p}_\xi'). \qquad (13)$$

These conditions can be formulated in any reference frame, if all momenta are replaced by

$$p_\perp := p - \frac{\langle p,P\rangle}{P^2}P\,. \qquad (14)$$

This space-like vector is perpendicular to the total four-momentum and in the rest frame of the system has the desired form $p_\perp = (0, \boldsymbol{p})$. Thus, formal covariance of the Bethe-Salpeter equation is maintained.

Adopting both approximations, it is possible to integrate out the dependence on the relative energies in the Bethe-Salpeter equation (8), thus reducing the eight-dimensional integral equation to the six-dimensional Salpeter equation. This procedure is straightforward if there are no two-body interactions in the system. The unconnected part of $\bar{K}^{(2)}$ however makes the reduction more involved. One is then forced to introduce an effective three-body kernel V_M^{eff} which accounts for the effect of the two-body interaction approximately (see [2] for details). The effective interaction kernel is then expanded in powers of $K_M^{(2)} + V_R^{(3)}$, where $V_R^{(3)}$ is the contribution to the three-body interaction $V^{(3)} = V_\Lambda^{(3)} + V_R^{(3)}$ that couples to mixed energy components exclusively. $V_\Lambda^{(3)}$ then correspondingly is the contribution to the three-particle kernel that involves only pure energy components. Up to the lowest-order Born approximation $V_M^{\mathrm{eff}\,(1)}$ the corresponding Salpeter equation then reads

$$\begin{aligned}
&\Phi_M^\Lambda(\boldsymbol{p}_\xi, \boldsymbol{p}_\eta) = \\
&\quad\left[\frac{\Lambda^{+++}}{M - \Omega + \mathrm{i}\epsilon} + \frac{\Lambda^{---}}{M + \Omega - \mathrm{i}\epsilon}\right] \gamma^0 \otimes \gamma^0 \otimes \gamma^0 \\
&\quad\times \int \frac{\mathrm{d}^3 p'_\xi}{(2\pi)^3} \frac{\mathrm{d}^3 p'_\eta}{(2\pi)^3} V^{(3)}(\boldsymbol{p}_\xi, \boldsymbol{p}_\eta; \boldsymbol{p}'_\xi, \boldsymbol{p}'_\eta) \Phi_M^\Lambda(\boldsymbol{p}'_\xi, \boldsymbol{p}'_\eta) \\
&\quad+ \left[\frac{\Lambda^{+++}}{M - \Omega + \mathrm{i}\epsilon} - \frac{\Lambda^{---}}{M + \Omega - \mathrm{i}\epsilon}\right] \gamma^0 \otimes \gamma^0 \otimes \mathbb{1} \\
&\quad\times \int \frac{\mathrm{d}^3 p'_\xi}{(2\pi)^3} V^{(2)}(\boldsymbol{p}_\xi, \boldsymbol{p}'_\xi) \otimes \mathbb{1}\, \Phi_M^\Lambda(\boldsymbol{p}'_\xi, \boldsymbol{p}_\eta) \\
&\quad+ \text{terms with cycl. perm. of two-body force.}
\end{aligned} \tag{15}$$

Here we introduced the short-hand notation $\Lambda^{\pm\pm\pm} := \Lambda_1^\pm(\boldsymbol{p}_1) \otimes \Lambda_2^\pm(\boldsymbol{p}_2) \otimes \Lambda_3^\pm(\boldsymbol{p}_3)$, where $\Lambda_i^\pm(\boldsymbol{p}_i)$ are projectors onto positive or negative energy, respectively and $\Omega := \omega_1(\boldsymbol{p}_1) + \omega_2(\boldsymbol{p}_2) + \omega_3(\boldsymbol{p}_3)$ is the sum of the relativistic one-particle energies $\omega_i(\boldsymbol{p}_i) = \sqrt{|\boldsymbol{p}_i|^2 + m_i^2}$. The Salpeter equation involves the Salpeter amplitudes, which are projected onto purely positive- and negative-energy components, respectively:

$$\begin{aligned}
\Phi_M^\Lambda(\boldsymbol{p}_\xi, \boldsymbol{p}_\eta) &:= \left[\Lambda^{+++}(\boldsymbol{p}_\xi, \boldsymbol{p}_\eta) + \Lambda^{---}(\boldsymbol{p}_\xi, \boldsymbol{p}_\eta)\right] \\
&\quad\times \int \frac{\mathrm{d}p_\xi^0}{2\pi} \frac{\mathrm{d}p_\eta^0}{2\pi} \chi_M(p_\xi, p_\eta).
\end{aligned} \tag{16}$$

The full Bethe-Salpeter amplitude, which is needed to calculate current matrix elements, can be reconstructed from the Salpeter amplitude in the following way:

$$\chi_M = \left[G_0 - \mathrm{i}G_0 \left(V_R^{(3)} + \bar{K}_M^{(2)} - V_P^{\mathrm{eff}\,(1)}\right) G_0\right] \Gamma_M^\Lambda, \tag{17}$$

where the so-called vertex functions Γ_M^Λ were introduced, which in lowest order in V_M^{eff} are connected to the Salpeter amplitudes by

$$\Phi_M^\Lambda = \mathrm{i} \left[\frac{\Lambda^{+++}}{M - \Omega} + \frac{\Lambda^{---}}{M + \Omega}\right] \gamma^0 \otimes \gamma^0 \otimes \gamma^0 \Gamma_M^\Lambda. \tag{18}$$

From the normalization condition (9) a corresponding normalization for the Salpeter amplitudes can be deduced, which in Born approximation reads

$$\langle \Phi_M^\Lambda | \Phi_M^\Lambda \rangle = \int \frac{\mathrm{d}^3 p_\xi}{(2\pi)^3} \frac{\mathrm{d}^3 p_\eta}{(2\pi)^3} \Phi_M^{\Lambda\,*}(\boldsymbol{p}_\xi, \boldsymbol{p}_\eta) \Phi_M^\Lambda(\boldsymbol{p}_\xi, \boldsymbol{p}_\eta) = 2M. \tag{19}$$

Summation over discrete indices is implicitly understood here. This norm immediately induces a positive definite scalar product:

$$\langle \Phi_1 | \Phi_2 \rangle := \int \frac{\mathrm{d}^3 p_\xi}{(2\pi)^3} \frac{\mathrm{d}^3 p_\eta}{(2\pi)^3} \Phi_1^*(\boldsymbol{p}_\xi, \boldsymbol{p}_\eta) \Phi_2(\boldsymbol{p}_\xi, \boldsymbol{p}_\eta), \tag{20}$$

whose existence is of utmost importance since static observables will be formulated as expectation values with respect to this scalar product as announced in the introduction.

2.3 Current matrix elements

To compute any electromagnetic observable, we need to know the electromagnetic current $\langle P, \lambda | j^\mu(x) | P', \lambda' \rangle$ between states with total four-momenta P' and P and helicities λ' and λ, respectively, where $j^\mu(x)$ is the current operator:

$$j^\mu(x) =: \bar{\psi}(x) \hat{q} \gamma^\mu \psi(x) :, \tag{21}$$

with the charge operator $\hat{q}$. This current matrix element can be derived by studying the response of the system in an external electromagnetic field in first order of the electromagnetic coupling strength [4]. One then finds that the corresponding correlation function $G_{P,P'}^\mu(p_\xi, p_\eta, p'_\xi, p'_\eta)$ separates at the poles in the total energy of the system in the following way:

$$\begin{aligned}
&G_{P,P'}^\mu(p_\xi, p_\eta, p'_\xi, p'_\eta) = \\
&\quad\frac{1}{4\omega_{\boldsymbol{P}}\omega'_{\boldsymbol{P}'}} \frac{\chi_P(p_\xi, p_\eta)}{P^0 - \omega_{\boldsymbol{P}} + \mathrm{i}\epsilon} \langle P | j^\mu(0) | P' \rangle \frac{\chi_{P'}(p_\xi, p_\eta)}{P'^0 - \omega_{\boldsymbol{P}'} + \mathrm{i}\epsilon} \\
&\qquad+ \text{regular terms for } P^0 \to \omega_{\boldsymbol{P}} \text{ and } P'^0 \to \omega_{\boldsymbol{P}'}.
\end{aligned} \tag{22}$$

The Mandelstam formalism and minimal coupling deliver an independent way to determine $G_{P,P'}^\mu$. By comparison one then finds the following current matrix element [4]:

$$\begin{aligned}
\langle P | j^\mu(0) | P' \rangle &= -3 \int \frac{\mathrm{d}^4 p_\xi}{(2\pi)^4} \int \frac{\mathrm{d}^4 p_\eta}{(2\pi)^4} \\
&\quad\times \overline{\Gamma}_P^\Lambda(p_\xi, p_\eta) S_F^1\big(p_\xi + \tfrac{1}{2}p_\eta\big) \otimes S_F^2\big(-p_\xi + \tfrac{1}{2}p_\eta\big) \\
&\quad\otimes S_F^3(P' - p_\eta) \gamma^\mu \hat{q} S_F^3(P - p_\eta) \Gamma_{P'}^\Lambda(p_\xi, p_\eta),
\end{aligned} \tag{23}$$

where the adjoint vertex function $\overline{\Gamma}_M^\Lambda(p_\xi, p_\eta)$ is related to $\Gamma_M^\Lambda(p_\xi, p_\eta)$ through

$$\overline{\Gamma}_M^\Lambda(p_\xi, p_\eta) = -\Gamma_M^{\Lambda\,\dagger}(p_\xi, p_\eta) \gamma^0 \otimes \gamma^0 \otimes \gamma^0. \tag{24}$$

Actually, the form of the matrix element in eq. (23) as calculated in ref. [4] involves the same kind of approximation as in eq. (17). For the following it is however important to note that this approximation respects the rules of formal covariance. Note that in the current matrix element (23) the photon couples to the third fermion exclusively. The couplings to the other fermions have been accounted for by a factor of 3. This is possible, since the vertex functions, which describe a composite fermion system, are totally antisymmetric. With the explicit boost prescription of the vertex function:

$$\Gamma_P^\Lambda(p_\xi, p_\eta) = S_{\Lambda_P} \otimes S_{\Lambda_P} \otimes S_{\Lambda_P} \Gamma_M^\Lambda(\boldsymbol{\Lambda}^{-1}\boldsymbol{p}_\xi, \boldsymbol{\Lambda}^{-1}\boldsymbol{p}_\eta), \quad (25)$$

the time component of the current matrix element in the Breit frame takes the form

$$\begin{aligned}\langle \mathcal{P}P|j^0(0)|P\rangle &= -3\int \frac{\mathrm{d}^4p_\xi}{(2\pi)^4}\int \frac{\mathrm{d}^4p_\eta}{(2\pi)^4}\overline{\Gamma}{}_M^\Lambda(\boldsymbol{p}_\xi, \boldsymbol{p}_\eta)\\ &\times \left[S_F^1\left(p_\xi + \tfrac{1}{2}p_\eta\right) \otimes S_F^2(-p_\xi + p_\eta) \otimes S_F^3(M - p_\eta)\right]\\ &\times \left[S_{\Lambda_P}^2 \otimes S_{\Lambda_P}^2 \otimes \mathbb{1}\right]\left[\mathbb{1} \otimes \mathbb{1} \otimes \gamma^0 \hat{q} S_F^3\left(M - {\Lambda_P^{-1}}^2 p_\eta\right)\right]\\ &\times \Gamma_M^\Lambda\left({\boldsymbol{\Lambda}_{\boldsymbol{P}}^{-1}}^2\boldsymbol{p}_\xi, {\boldsymbol{\Lambda}_{\boldsymbol{P}}^{-1}}^2\boldsymbol{p}_\eta\right), \end{aligned} \quad (26)$$

where $\mathcal{P}$ is the space inversion operator, *i.e.* $\mathcal{P}(x^0, \boldsymbol{x}) = (x^0, -\boldsymbol{x})$. The spatial components are accordingly

$$\begin{aligned}\langle \mathcal{P}P|j^i(0)|P\rangle &= -3\int \frac{\mathrm{d}^4p_\xi}{(2\pi)^4}\int \frac{\mathrm{d}^4p_\eta}{(2\pi)^4}\overline{\Gamma}{}_M^\Lambda(\boldsymbol{p}_\xi, \boldsymbol{p}_\eta)\\ &\times \left[S_F^1\left(p_\xi + \tfrac{1}{2}p_\eta\right) \otimes S_F^2(-p_\xi + p_\eta) \otimes S_F^3(M - p_\eta)\right]\\ &\times \left[S_{\Lambda_P}^2 \otimes S_{\Lambda_P}^2 \otimes S_{\Lambda_P}^2\right]\\ &\times \left[\mathbb{1} \otimes \mathbb{1} \otimes \hat{q}\left(\gamma^i + [\gamma^i, S_{\Lambda_P}]_-\right) S_F^3\left(M - {\Lambda_P^{-1}}^2 p_\eta\right)\right]\\ &\times \Gamma_M^\Lambda\left({\boldsymbol{\Lambda}_{\boldsymbol{P}}^{-1}}^2\boldsymbol{p}_\xi, {\boldsymbol{\Lambda}_{\boldsymbol{P}}^{-1}}^2\boldsymbol{p}_\eta\right). \end{aligned} \quad (27)$$

Note that in both cases we commuted a triple tensor product of S_{Λ_P} past the three fermion propagators and then did an integral transformation to obtain two successive boosts. This also explains the appearance of the commutator $[\gamma^i, S_{\Lambda_P}]_-$ in the spatial components of the current matrix element.

3 The charge radius

3.1 From charge distributions to charge radii

In the next two section we review the definition and precise computation of a particular observable namely the charge radius of a composite system in the framework of quantum field theory. The results are well known but we add them here for a better understanding. Given some charge distribution $\rho(\boldsymbol{x})$ one defines its mean square radius by

$$\langle r^2\rangle = \frac{1}{Q}\int \mathrm{d}^3x\, |\boldsymbol{x}|^2 \rho(\boldsymbol{x}). \quad (28)$$

The radius is normalized by the net charge Q, which is simply the integral of $\rho(\boldsymbol{x})$ over the whole space:

$$Q = \int \mathrm{d}^3x\, \rho(\boldsymbol{x}). \quad (29)$$

However, if the charge distribution has no net charge, the normalization $1/Q$ is of course dropped. If we turn to quantum-mechanical systems, the charge distribution is given by the time component $j^0(x)$ of the four-vector current of the state $|\psi\rangle$ that describes the system:

$$\rho(\boldsymbol{x}) = \frac{\langle\psi|j^0(\boldsymbol{x})|\psi\rangle}{\langle\psi|\psi\rangle}. \quad (30)$$

Such a state $|\psi\rangle$ can be represented as a superposition of momentum eigenstates

$$|\psi\rangle = \int \frac{\mathrm{d}^3P}{\omega_{\boldsymbol{P}}}\psi(\boldsymbol{P})|P\rangle. \quad (31)$$

$\psi(\boldsymbol{P})$ is the wave function in momentum space and the states $|P\rangle$ are normalized according to

$$\langle P|P'\rangle = 2\omega_{\boldsymbol{P}}(2\pi)^3\delta^{(3)}(\boldsymbol{P} - \boldsymbol{P}'). \quad (32)$$

This immediately fixes the normalization of the states $|\psi\rangle$:

$$\langle\psi|\psi\rangle = 2(2\pi)^3\int \frac{\mathrm{d}^3P}{\omega_{\boldsymbol{P}}}\psi^*(\boldsymbol{P})\psi(\boldsymbol{P}). \quad (33)$$

Let us further investigate the charge distribution by inserting (31) into (30):

$$\begin{aligned}\rho(\boldsymbol{x}) = \frac{1}{\langle\psi|\psi\rangle}\int \frac{\mathrm{d}^3P}{\omega_{\boldsymbol{P}}}\int \frac{\mathrm{d}^3P'}{\omega_{\boldsymbol{P}'}}\exp\left(\mathrm{i}(\boldsymbol{P} - \boldsymbol{P}')\cdot\boldsymbol{x}\right)\\ \times\psi^*(\boldsymbol{P})\psi(\boldsymbol{P}')\langle P|j^0(0)|P'\rangle. \end{aligned} \quad (34)$$

We used space translation invariance here to separate the spatial dependence. As is well known the integral $\int \mathrm{d}^3x \exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x})$ is a representation of the delta distribution. So it follows:

$$\begin{aligned}&\int \mathrm{d}^3x\, |\boldsymbol{x}|^2 \exp\left(\mathrm{i}\,(\boldsymbol{P} - \boldsymbol{P}')\cdot\boldsymbol{x}\right)\\ &\qquad = -\frac{(2\pi)^3}{4}(\boldsymbol{\nabla}_{\boldsymbol{P}} - \boldsymbol{\nabla}_{\boldsymbol{P}'})^2\,\delta^{(3)}(\boldsymbol{P} - \boldsymbol{P}'). \end{aligned} \quad (35)$$

Using (31), (34) and (35) we obtain

$$\begin{aligned}\int \mathrm{d}^3x\, |\boldsymbol{x}|^2\langle\psi|j^0(x)|\psi\rangle = -\frac{(2\pi)^3}{4}\int \frac{\mathrm{d}^3P}{\omega_{\boldsymbol{P}}}\int \frac{\mathrm{d}^3P'}{\omega_{\boldsymbol{P}'}}\\ \times\psi^*(\boldsymbol{P})\psi(\boldsymbol{P}')\langle P|j^0(0)|P'\rangle\\ \times(\boldsymbol{\nabla}_{\boldsymbol{P}} - \boldsymbol{\nabla}_{\boldsymbol{P}'})^2\delta^{(3)}(\boldsymbol{P} - \boldsymbol{P}'). \end{aligned} \quad (36)$$

On the right-hand side of this equation we may now integrate by parts twice and subsequently do one of the two momentum integrations:

$$\begin{aligned}&\int \mathrm{d}^3x\, |\boldsymbol{x}|^2\langle\psi|j^0(x)|\psi\rangle = -\frac{(2\pi)^3}{4}\int \mathrm{d}^3P\\ &\times\Bigg\{\frac{|\psi(\boldsymbol{P})|^2}{\omega_{\boldsymbol{P}}^2}\left[(\boldsymbol{\nabla}_{\boldsymbol{P}} - \boldsymbol{\nabla}_{\boldsymbol{P}'})^2\langle P|j^0(0)|P'\rangle\right]_{\boldsymbol{P}'=\boldsymbol{P}}\\ &-\left[(\boldsymbol{\nabla}_{\boldsymbol{P}} - \boldsymbol{\nabla}_{\boldsymbol{P}'})^2\frac{\psi^*(\boldsymbol{P})}{\omega_{\boldsymbol{P}}}\frac{\psi(\boldsymbol{P}')}{\omega_{\boldsymbol{P}'}}\right]_{\boldsymbol{P}'=\boldsymbol{P}}\langle P|j^0(0)|P\rangle\\ &-\left[(\boldsymbol{\nabla}_{\boldsymbol{P}} - \boldsymbol{\nabla}_{\boldsymbol{P}'})\frac{\psi^*(\boldsymbol{P})}{\omega_{\boldsymbol{P}}}\frac{\psi(\boldsymbol{P}')}{\omega_{\boldsymbol{P}'}}\right]_{\boldsymbol{P}'=\boldsymbol{P}}\\ &\cdot\left[(\boldsymbol{\nabla}_{\boldsymbol{P}} - \boldsymbol{\nabla}_{\boldsymbol{P}'})\langle P|j^0(0)|P'\rangle\right]_{\boldsymbol{P}'=\boldsymbol{P}}\Bigg\}. \end{aligned} \quad (37)$$

The last of this three terms vanishes, because $\boldsymbol{\nabla}_{\boldsymbol{P}}$ and $\boldsymbol{\nabla}_{\boldsymbol{P}'}$ change sign under space reflection, in other words are of odd parity. So if we assume that the states $|P\rangle$ have definite parity then $\langle P|j^0(0)\boldsymbol{\nabla}_{\boldsymbol{P}}|P\rangle = 0$.

So far we have considered wave packets that consist of a superposition of states with different momenta. To obtain states with definite *i.e.* sharp momenta consider first a Gaussian wave packet with a width proportional to some parameter λ:

$$\psi(\boldsymbol{P}) = \exp(-|\boldsymbol{P}|^2/(2\lambda^2)). \tag{38}$$

Before we let λ go to zero to define a definite momentum state let us inspect the second term in (37). Because for the Gaussian wave packet from (38) the wave function is real, *i.e.* $\psi^*(\boldsymbol{P}) = \psi(\boldsymbol{P})$, we have

$$\left[(\boldsymbol{\nabla}_{\boldsymbol{P}} - \boldsymbol{\nabla}_{\boldsymbol{P}'})^2 \frac{\psi(\boldsymbol{P})}{\omega_{\boldsymbol{P}}} \frac{\psi(\boldsymbol{P}')}{\omega_{\boldsymbol{P}'}}\right]_{\boldsymbol{P}'=\boldsymbol{P}} = 0. \tag{39}$$

Therefore also this term does not contribute to the charge radius and we are left with the first term in (37) only.

Let us now turn to the limit $\lambda \to 0$ again. Since in this limit $\exp(-|\boldsymbol{P}|^2/\lambda^2)$ is another representation of the delta distribution we find

$$\lim_{\lambda\to 0} \frac{|\psi(\boldsymbol{P})|^2}{\omega_{\boldsymbol{P}}\langle\psi(\boldsymbol{P})|\psi(\boldsymbol{P})\rangle} = \frac{\delta^{(3)}(\boldsymbol{P})}{2(2\pi)^3}\,. \tag{40}$$

Inserting this together with the first term of (37) into the basic definition of the charge radius (28) and performing the final momentum integration yields

$$\langle r^2\rangle = -\frac{1}{8MQ}(\boldsymbol{\nabla}_{\boldsymbol{P}} - \boldsymbol{\nabla}_{\boldsymbol{P}'})^2\langle P|j^0(0)|P'\rangle\bigg|_{\boldsymbol{P}'=\boldsymbol{P}=0}, \tag{41}$$

where M is the rest mass of the system. The current matrix element appearing here is given in the Breit frame if we make the following transformation:

$$(\boldsymbol{\nabla}_{\boldsymbol{P}} - \boldsymbol{\nabla}_{\boldsymbol{P}'})^2\langle P|j^0(0)|P'\rangle\bigg|_{\boldsymbol{P}'=\boldsymbol{P}=0} = \Delta_{\boldsymbol{P}}\langle \mathcal{P}P|j^0(0)|P\rangle\bigg|_{\boldsymbol{P}=0}. \tag{42}$$

We then finally end up with the expression

$$\langle r^2\rangle = -\frac{1}{8MQ}\Delta_{\boldsymbol{P}}\langle \mathcal{P}P|j^0(0)|P\rangle\bigg|_{\boldsymbol{P}=0}. \tag{43}$$

3.2 From form factors to charge radii

In the last section the derivation of the charge radius operator started from defining the mean square radius of a charge distribution. As is well known there is another definition of the charge radius that involves the electric form factor of the system. There the charge radius is defined as the slope of the electric form factor at the photon point. In this subsection we investigate the interconnection between both definitions and show that they indeed coincide.

Let us briefly recall some basic definitions in the context of form factors. From current conservation and Lorentz invariance the electromagnetic vector current of a spin-1/2 state can be parameterized as follows:

$$\langle P',\lambda'|j^\mu(0)|P,\lambda\rangle = e\bar{u}_{\lambda'}(P') \times \left[\gamma^\mu\left(F_1(Q^2)+F_2(Q^2)\right) - \frac{P'^\mu + P^\mu}{2M}F_2(Q^2)\right]u_\lambda(P). \tag{44}$$

F_1 and F_2 are the Dirac and Pauli form factors, respectively. The Dirac form factor is normalized to the charge Q, whereas the Pauli form factor is normalized to the anomalous magnetic moment κ. Both form factors are functions of the squared invariant momentum transfer $Q^2 := -q^2 = -(P'-P)^2$. The Dirac spinors are normalized in a Lorentz-invariant fashion:

$$\bar{u}_{\lambda'}(P)u_\lambda(P) = 2M\,\delta_{\lambda'\lambda}. \tag{45}$$

Using this normalization one shows that

$$\bar{u}_{\lambda'}(\mathcal{P}P)u_\lambda(P) = 2\sqrt{M^2+Q^2/4}\,\delta_{\lambda'\lambda} \tag{46}$$

as well as

$$\bar{u}_{\lambda'}(\mathcal{P}P)\gamma^0u_\lambda(P) = 2M\delta_{\lambda'\lambda}. \tag{47}$$

Using both expressions one can write the time component of the electromagnetic vector current (44) in the Breit frame as

$$\langle \mathcal{P}P,\lambda|j^0(0)|P,\lambda\rangle = 2eMG_E(Q^2), \tag{48}$$

where $G_E(Q^2)$ is the electric Sachs form factor. It is defined together with the magnetic Sachs form factor as a combination of the Dirac and Pauli form factors:

$$G_E(Q^2) := F_1(Q^2) - \frac{Q^2}{4M^2}F_2(Q^2), \tag{49}$$

$$G_M(Q^2) := F_1(Q^2) + F_2(Q^2). \tag{50}$$

The mean square charge radius is defined as the slope of the electric form factor at the photon point *i.e.* at $Q^2 = 0$:

$$\langle r^2\rangle = -\frac{6}{G_E(0)}\frac{\mathrm{d}G_E(Q^2)}{\mathrm{d}Q^2}\bigg|_{Q^2=0}. \tag{51}$$

Since $G_E(0) = F_1(0) = Q$ the normalization $1/G_E(0)$ is dropped in case the net charge vanishes. From this definition together with (48) we then find

$$\langle r^2\rangle = -\frac{3}{MQ}\frac{\mathrm{d}}{\mathrm{d}Q^2}\langle \mathcal{P}P,\lambda|j^0(0)|P,\lambda\rangle\bigg|_{Q^2=0}. \tag{52}$$

This result has to be compared to the one that we obtained in the previous section, namely eq. (43). There we found the Laplace operator with respect to $\boldsymbol{P}$ instead of a single derivative with respect to Q^2 acting on the current matrix element. However both expressions turn out to be exactly equal: From the parameterization of the vector current (44) it is clear that the current matrix element $\langle \mathcal{P}P,\lambda|j^0(0)|P,\lambda\rangle$ depends on Q^2. In the Breit frame the dependence of Q^2 on the momenta of the incoming and outgoing bound states becomes rather simple. It reads $Q^2 = 4|\boldsymbol{P}|^2$. Then, for any function f depending on $4|\boldsymbol{P}|^2$ the following identity holds:

$$\Delta_{\boldsymbol{P}} f\left(4|\boldsymbol{P}|^2\right) = 4\left(\Delta_{\boldsymbol{P}}|\boldsymbol{P}|^2\right)\frac{\mathrm{d}}{\mathrm{d}Q^2}f(Q^2) = 24\frac{\mathrm{d}}{\mathrm{d}Q^2}f(Q^2). \tag{53}$$

Inserting this into (43) we see that it coincides with (52). Thus, the definition of the charge radius from form factors is exactly equivalent to that from charge distributions. It must be noted, however, that expression (43) is somewhat more general than (52) in the sense that it is valid for particles with arbitrary spin. Nevertheless, one can show that both expressions lead to the same result. We decided, however, to start from (52) simply because it contains only a first-order derivative.

3.3 The charge radius as an expectation value with respect to Salpeter amplitudes

We now want to analyze the charge radius in the Bethe-Salpeter framework for a three-quark system. It is essential to know the Q^2-dependence of the current matrix element (26). So let us inspect its P-dependent part alone:

$$\begin{aligned}&\left[S^2_{\Lambda_P}\otimes S^2_{\Lambda_P}\otimes S^3_F\left(M-{\Lambda_P^{-1}}^2 p_\eta\right)\right]\Gamma^\Lambda_M(\overrightarrow{{\Lambda_P^{-1}}^2 p_\xi},\overrightarrow{{\Lambda_P^{-1}}^2 p_\eta})\\ &\quad := \left[S^2_{\Lambda_P}\otimes S^2_{\Lambda_P}\otimes \mathbb{1}\right] f\left({\Lambda_P^{-1}}^2 p_\xi, {\Lambda_P^{-1}}^2 p_\eta\right).\end{aligned} \tag{54}$$

We now exploit an important property of Lie groups, namely that every group element may be represented as an exponential mapping of the Lie algebra:

$$\begin{aligned}&\left[S^2_{\Lambda_P}\otimes S^2_{\Lambda_P}\otimes \mathbb{1}\right] f\left({\Lambda_P^{-1}}^2 p_\xi, {\Lambda_P^{-1}}^2 p_\eta\right)\\ &\qquad = \exp(-2\mathrm{i}\,\boldsymbol{\eta}(P)\cdot\hat{\boldsymbol{K}})f(p_\xi,p_\eta).\end{aligned} \tag{55}$$

The parameter $\boldsymbol{\eta}$, commonly called rapidity, is defined as follows:

$$\boldsymbol{\eta}(P) := \frac{\boldsymbol{P}}{P^0} = \frac{-\boldsymbol{q}}{2\sqrt{M^2+Q^2/4}}, \tag{56}$$

where the last equality follows from Breit frame kinematics. The operator $\hat{\boldsymbol{K}}$ is an infinitesimal boost. The generators of the Lorentz group are given by the following skew symmetric tensors (see, *e.g.* [6]):

$$J^{\mu\nu} = \mathrm{i}(x^\mu\partial^\nu - x^\nu\partial^\mu), \tag{57}$$

$$S^{\mu\nu} = \tfrac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]_-, \tag{58}$$

Because of skewness there are six independent quantities. J^{0i} are the three generators of boosts and the remaining three operators generate rotations (in fact they are the angular-momentum operators). In momentum space we have $J^{\mu\nu}_p = \mathrm{i}(p^\mu\partial/\partial p_\nu - p^\nu\partial/\partial p_\mu)$. $S^{\mu\nu}$ are the corresponding generators in Dirac space. The infinitesimal boost $\hat{\boldsymbol{K}}$ then simply reads

$$\begin{aligned}\hat{K}^i &= -J^{0i}_{p_\xi} - J^{0i}_{p_\eta} + S^{0i}\otimes\mathbb{1}\otimes\mathbb{1} + \mathbb{1}\otimes S^{0i}\otimes\mathbb{1}\\ &= \mathrm{i}\big(-p^0_\xi\tfrac{\partial}{\partial p^i_\xi} - p^i_\xi\tfrac{\partial}{\partial p^0_\xi} - p^0_\eta\tfrac{\partial}{\partial p^i_\eta} - p^i_\eta\tfrac{\partial}{\partial p^0_\eta}\\ &\quad + \tfrac12\alpha^i\otimes\mathbb{1}\otimes\mathbb{1} + \mathbb{1}\otimes\tfrac12\alpha^i\otimes\mathbb{1}\big).\end{aligned} \tag{59}$$

Inserting (55) back into the current matrix element (26) we find

$$\begin{aligned}\langle\mathcal{P}P|j^0(0)|P\rangle &= -3\int\frac{\mathrm{d}^4p_\xi}{(2\pi)^4}\int\frac{\mathrm{d}^4p_\eta}{(2\pi)^4}\overline{\Gamma}^\Lambda_M(\boldsymbol{p}_\xi,\boldsymbol{p}_\eta)\\ &\times\left[S^1_F\left(p_\xi+\tfrac12 p_\eta\right)\otimes S^2_F(-p_\xi+p_\eta)\otimes S^3_F(M-p_\eta)\right]\\ &\times\exp(-2\mathrm{i}\,\boldsymbol{\eta}(P)\cdot\hat{\boldsymbol{K}})\left[\mathbb{1}\otimes\mathbb{1}\otimes\gamma^0\hat{q}S^3_F(M-p_\eta)\right]\\ &\times\Gamma^\Lambda_M(p_\xi,p_\eta).\end{aligned} \tag{60}$$

Since the charge radius is proportional to the slope of the current matrix element (60) at $Q^2=0$ we are interested in the term of the expansion linear in Q^2 and thus —because $|\boldsymbol{\eta}(P)|^2$ is of order Q^2— linear in $|\boldsymbol{\eta}(P)|^2$. Writing the expansion explicitly out up to this order we have

$$\begin{aligned}&\langle\mathcal{P}P|j^0(0)|P\rangle = 2MQ\\ &-3\int\frac{\mathrm{d}^4p_\xi}{(2\pi)^4}\int\frac{\mathrm{d}^4p_\eta}{(2\pi)^4}\overline{\Gamma}^\Lambda_M(\boldsymbol{p}_\xi,\boldsymbol{p}_\eta)\\ &\times\left[S^1_F\left(p_\xi+\tfrac12 p_\eta\right)\otimes S^2_F(-p_\xi+p_\eta)\otimes S^3_F(M-p_\eta)\right]\\ &\times\left[-2\sum_{i,j=1}^{3}\eta^i(P)\eta^j(P)\hat{K}^i\hat{K}^j\right]\\ &\times\left[\mathbb{1}\otimes\mathbb{1}\otimes\gamma^0\hat{q}S^3_F(M-p_\eta)\right]\Gamma^\Lambda_M(p_\xi,p_\eta)+\mathcal{O}(\boldsymbol{\eta}^4).\end{aligned} \tag{61}$$

By inserting this into (52), the charge radius then takes the form:

$$\begin{aligned}\langle r^2\rangle &= -\frac{18}{MQ}\sum_{i,j=1}^{3}\left[\frac{\mathrm{d}}{\mathrm{d}Q^2}\eta^i(P)\eta^j(P)\right]_{Q^2=0}\\ &\times\int\frac{\mathrm{d}^4p_\xi}{(2\pi)^4}\int\frac{\mathrm{d}^4p_\eta}{(2\pi)^4}\overline{\Gamma}^\Lambda_M(\boldsymbol{p}_\xi,\boldsymbol{p}_\eta)\\ &\times\left[S^1_F\left(p_\xi+\tfrac12 p_\eta\right)\otimes S^2_F(-p_\xi+p_\eta)\otimes S^3_F(M-p_\eta)\right]\hat{K}^i\hat{K}^j\\ &\times\left[\mathbb{1}\otimes\mathbb{1}\otimes\gamma^0\hat{q}S^3_F(M-p_\eta)\right]\Gamma^\Lambda_M(p_\xi,p_\eta).\end{aligned} \tag{62}$$

The integration over the relative energies can now be performed by using the partial fractions decomposition of the fermion propagators:

$$S^i_F(p_i) = \mathrm{i}\left(\frac{\Lambda^+_i(\boldsymbol{p}_i)}{p^0_i-\omega_i(\boldsymbol{p}_i)+\mathrm{i}\epsilon}+\frac{\Lambda^-_i(\boldsymbol{p}_i)}{p^0_i+\omega_i(\boldsymbol{p}_i)-\mathrm{i}\epsilon}\right)\gamma^0. \tag{63}$$

With the aid of Cauchys theorem both integrations can be performed and one obtains

$$\begin{aligned}\langle r^2\rangle &= \frac{18}{MQ}\sum_{i,j=1}^{3}\left[\frac{\mathrm{d}}{\mathrm{d}Q^2}\eta^i(P)\eta^j(P)\right]_{Q^2=0}\\ &\times\int\frac{\mathrm{d}^3p_\xi}{(2\pi)^3}\int\frac{\mathrm{d}^3p_\eta}{(2\pi)^3}\overline{\Gamma}^\Lambda_M(\boldsymbol{p}_\xi,\boldsymbol{p}_\eta)\left[\frac{\Lambda^{+++}}{(M-\Omega)}+\frac{\Lambda^{---}}{(M+\Omega)}\right]\\ &\times\hat{K}'^i\hat{K}'^j\hat{q}^3\left[\frac{\Lambda^{+++}}{(M-\Omega)}+\frac{\Lambda^{---}}{(M+\Omega)}\right]\\ &\times[\gamma^0\otimes\gamma^0\otimes\gamma^0]\Gamma^\Lambda_M(\boldsymbol{p}_\xi,\boldsymbol{p}_\eta). \end{aligned}\tag{64}$$

Now $\hat{q}^3$ denotes the charge operator acting on the third fermion. After integration the boost becomes

$$\hat{K}'^i := -\tfrac{1}{2}(\omega_1-\omega_2)\mathrm{i}\tfrac{\partial}{\partial p^i_\xi}-(\omega_1+\omega_2)\mathrm{i}\tfrac{\partial}{\partial p^i_\eta}-\tfrac{\mathrm{i}p^i_1}{2\omega_1}-\tfrac{\mathrm{i}p^i_2}{2\omega_2}. \tag{65}$$

Note that we used the anticommutator $\{\gamma^0,\alpha^i\}_+=0$ and the relation $\Lambda^\pm_i\alpha^j=\alpha^j\Lambda^\mp_i\pm p^j_i/\omega_i$ here. Now by using relations (18) and (24) one can replace the vertex functions in (64) by Salpeter amplitudes. The result is an expectation value with respect to the Salpeter scalar product (20):

$$\begin{aligned}\langle r^2\rangle &= \frac{18}{MQ}\sum_{i,j=1}^{3}\left[\frac{\mathrm{d}}{\mathrm{d}Q^2}\eta^i(P)\eta^j(P)\right]_{Q^2=0}\\ &\times\langle\Phi^\Lambda_M|\hat{K}'^i\hat{K}'^j\hat{q}^3|\Phi^\Lambda_M\rangle,\end{aligned}\tag{66}$$

Since $\hat{\boldsymbol{K}}'$ is a tensor operator of rank 1, that is a vector operator, $\hat{K}'^i\hat{K}'^j$ is a Cartesian tensor operator of rank 2. As is well known, every Cartesian tensor may be decomposed into irreducible representations of the rotation group $SO(3)$. The decomposition of a rank-2 tensor T_{ij} is given by

$$\begin{aligned}T_{ij} &= \tfrac{1}{3}\,\mathrm{tr}(T)\delta_{ij}+\tfrac{1}{2}(T_{ij}-T_{ji})\\ &+\tfrac{1}{2}(T_{ij}+T_{ji}-\tfrac{2}{3}\,\mathrm{tr}(T)\delta_{ij}).\end{aligned}\tag{67}$$

According to their transformation properties under rotations, the first term belongs to the scalar representation, the second to the vector representation and the last to the five-dimensional representation of spin 2. Let us now address the question, which of these representations will vanish due to selection rules in the scalar product in (66). Let us start with the vector representation:

$$\begin{aligned}&\sum_{i,j=1}^{3}\left[\frac{\mathrm{d}}{\mathrm{d}Q^2}\eta^i(P)\eta^j(P)\right]_{Q^2=0}\\ &\quad\times\langle\Phi^\Lambda_M|\tfrac{1}{2}\left(\hat{K}'^i\hat{K}'^j-\hat{K}'^j\hat{K}'^i\right)\hat{q}^3|\Phi^\Lambda_M\rangle=0.\end{aligned}\tag{68}$$

This is so because $\eta^i(P)\eta^j(P)$ is symmetric, whereas $\hat{K}'^i\hat{K}'^j-\hat{K}'^j\hat{K}'^i$ is antisymmetric under the exchange of indices. For the spin-2 representation we cite the Wigner-Eckart theorem and in particular the triangularity relation which states that for spherical tensor operators of rank k,

$$\langle j_1|T^{[k]}_q|j_2\rangle=0\quad\text{unless}\quad|j_1-j_2|\le k\le j_1+j_2. \tag{69}$$

In our case $j_1=j_2=\frac{1}{2}$ and $k=2$, so the spin-2 representation in (67) gives no contribution. Only the scalar representation contributes and we get from (67) and (66)

$$\langle r^2\rangle=\frac{6}{MQ}\left[\frac{\mathrm{d}}{\mathrm{d}Q^2}\boldsymbol{\eta}^2(P)\right]_{Q^2=0}\langle\Phi^\Lambda_M|\hat{\boldsymbol{K}}'^2\hat{q}^3|\Phi^\Lambda_M\rangle. \tag{70}$$

Recalling the definition of the rapidity (56) we find

$$\left.\frac{\mathrm{d}}{\mathrm{d}Q^2}\boldsymbol{\eta}^2(P)\right|_{Q^2=0}=\frac{1}{4M^2}\,, \tag{71}$$

which brings us almost to our final result:

$$\langle r^2\rangle=\frac{3}{2MQ}\langle\Phi^\Lambda_M|\frac{\hat{\boldsymbol{K}}'^2}{M^2}\hat{q}^3|\Phi^\Lambda_M\rangle, \tag{72}$$

By rewriting $\hat{K'}^i$ in terms of one-particle coordinates, we find

$$\hat{K}'^i=\frac{1}{2}\left[\Omega\left(\mathrm{i}\tfrac{\partial}{\partial p^i_3}-\frac{1}{\Omega}\sum_{\alpha=1}^{3}\omega_\alpha\mathrm{i}\tfrac{\partial}{\partial p^i_\alpha}\right)+\text{h.c.}\right]. \tag{73}$$

It is also useful to define

$$\hat{\boldsymbol{R}}:=\frac{1}{\Omega}\sum_{\alpha=1}^{3}\omega_\alpha\mathrm{i}\boldsymbol{\nabla}_{p_\alpha}. \tag{74}$$

The expression (72) is still not symmetric in all three particles. The third fermion seems to play a special role. However, this asymmetry is only due to the fact that in deriving the current matrix element we exploited the total asymmetry of the vertex functions under particle interchange and coupled the photon to the third fermion exclusively accounting for the other couplings by multiplying with a factor of 3 (see [4]). If we reverse this procedure, cancel the factor of 3 and symmetrize the expression over the three particles, we end up with a symmetric version:

$$\begin{aligned}\langle r^2\rangle &= \frac{1}{Q\,\langle\Phi^\Lambda_M|\Phi^\Lambda_M\rangle}\\ &\times\langle\Phi^\Lambda_M|\sum_{\alpha=1}^{3}\frac{\hat{q}^\alpha}{4}\left[\frac{\Omega}{M}\left(\mathrm{i}\boldsymbol{\nabla}_{p_\alpha}-\hat{\boldsymbol{R}}\right)+\text{h.c.}\right]^2|\Phi^\Lambda_M\rangle.\end{aligned}\tag{75}$$

Now the sum runs over all three fermions. Note that we also used the norm of the Salpeter amplitudes (19) to replace $2M$ by $\langle\Phi^\Lambda_M|\Phi^\Lambda_M\rangle$.

3.4 Interpretation

Having derived an analytic expression for the mean square charge radius of a bound three-fermion system with instantaneous interaction kernels, it is worthwhile to give the result a meaningful physical interpretation. To interpret the operator between the Salpeter amplitudes it is useful to note that $\mathrm{i}\boldsymbol{\nabla}_{p_\alpha}$ is the position operator in momentum space:

$$\mathrm{i}\boldsymbol{\nabla}_{p_\alpha}\equiv\hat{\boldsymbol{x}}_\alpha. \tag{76}$$

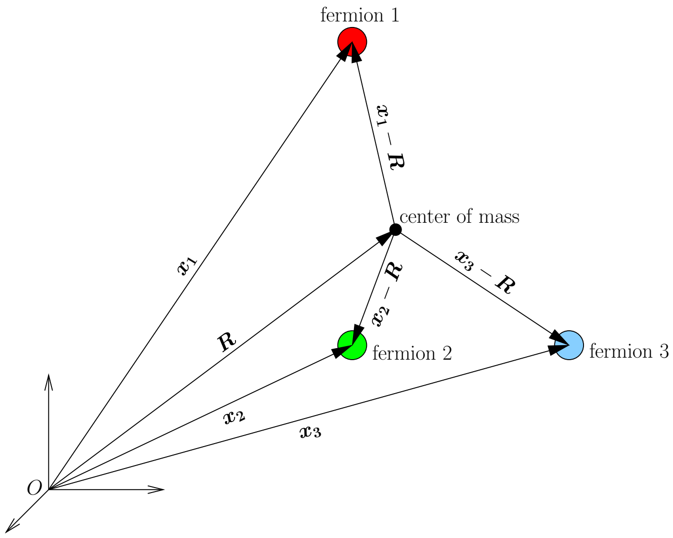

Fig. 1. O denotes the origin of the reference frame, $\boldsymbol{R}$ the position vector of the relativistic center of mass and $\boldsymbol{x}_1$ through $\boldsymbol{x}_3$ the position vectors of the three fermions.

Consequently, the quantity $\hat{\boldsymbol{R}}$ as defined in (74) is the canonical relativistic center of mass of a three-particle system:

$$\hat{\boldsymbol{R}} = \frac{1}{\Omega}\sum_{\alpha=1}^{3} \omega_\alpha \hat{\boldsymbol{x}}_\alpha. \tag{77}$$

At fermion momenta small compared to their masses, *i.e.* $|\boldsymbol{p}_\alpha| \ll m_\alpha$, we have $\omega_\alpha \to m_\alpha$ and thus the expression reduces to the well-known non-relativistic center of mass:

$$\hat{\boldsymbol{R}}_{\mathrm{nr}} = \frac{1}{m_1 + m_2 + m_3}\sum_{\alpha=1}^{3} m_\alpha \hat{\boldsymbol{x}}_\alpha. \tag{78}$$

The expression

$$\mathrm{i}\boldsymbol{\nabla}_{p_\alpha} - \hat{\boldsymbol{R}} = \hat{\boldsymbol{x}}_\alpha - \frac{1}{\Omega}\sum_{\beta=1}^{3} \omega_\beta \hat{\boldsymbol{x}}_\beta \tag{79}$$

then corresponds to the position of particle α as measured from the relativistic center of mass. Figure 1 illustrates the situation. Since $\mathrm{i}\boldsymbol{\nabla}_{p_\alpha} - \hat{\boldsymbol{R}}$ is the difference between two vector operators, it is invariant under translations and consequently the mean square radius (75) is translationally invariant. Finally, we want to call attention to the relativistic factor Ω/M in (75) which weights the relative distance of each fermion with the collective relativistic energy. Therefore an enhancement can be seen the more relativistic the system is.

4 The magnetic moment

4.1 The magnetic moment as an expectation value

To derive magnetic moments from form factors in an analogous way we start from the parameterization (44) of the electromagnetic vector current:

$$\begin{aligned}\langle \mathcal{P}P, \lambda' | j^+(0) | P, \lambda\rangle \\ = e\left[F_1(Q^2) + F_2(Q^2)\right] \bar{u}_{\lambda'}(\mathcal{P}P)\gamma^+ u_\lambda(P),\end{aligned} \tag{80}$$

where the "+"-component of the current is defined by:

$$j^+(0) = j^1(0) + \mathrm{i}j^2(0). \tag{81}$$

Note that this definition deviates from the definition of the components of a spherical tensor operator of rank 1 which are usually given by

$$T^{[1]}_\pm := \mp\frac{1}{\sqrt{2}}(T_1 \pm \mathrm{i}T_2) \quad \text{and} \quad T^{[1]}_0 := T_3. \tag{82}$$

The total spin of the system makes of course a spin flip so we have $\lambda' \neq \lambda$. The spin polarizations will be fixed later. Evaluation of the spinorial part of this equation yields

$$\bar{u}_{\lambda'}(\mathcal{P}P)\gamma^+ u_\lambda(P) = 2\sqrt{Q^2}\,. \tag{83}$$

Together with the definition of the magnetic Sachs form factor (49) we then get from (80) the relation

$$G_M(Q^2) = \frac{\langle \mathcal{P}P, \lambda' | j^+(0) | P, \lambda\rangle}{2\sqrt{Q^2}}\,, \tag{84}$$

which expresses the magnetic form factor in terms of spatial components of the current matrix element. The magnetic moment is defined as the value of the magnetic form factor at the photon point

$$\langle \mu \rangle := G_M(Q^2 = 0). \tag{85}$$

Because of the denominator in (84) taking this limit requires some care. We need to know the Q^2-dependence of the current matrix element. But first let us choose the three-momentum transfer to point in the 3-direction from now on:

$$\boldsymbol{q} := \begin{pmatrix} 0 \\ 0 \\ q_3 \end{pmatrix} = \sqrt{Q^2}\,\boldsymbol{e}_3. \tag{86}$$

With this choice we have $[\gamma^+, S_{\Lambda_P}]_- = 0$ such that the current matrix element (27) now becomes

$$\begin{aligned}\langle \mathcal{P}P | j^+(0) | P\rangle &= -3\int \frac{\mathrm{d}^4 p_\xi}{(2\pi)^4}\int \frac{\mathrm{d}^4 p_\eta}{(2\pi)^4} \overline{\Gamma}^\Lambda_M(\boldsymbol{p}_\xi, \boldsymbol{p}_\eta) \\ &\times \left[S^1_F\left(p_\xi + \tfrac{1}{2}p_\eta\right) \otimes S^2_F(-p_\xi + p_\eta) \otimes S^3_F(M - p_\eta)\right] \\ &\times \left[S^2_{\Lambda_P} \otimes S^2_{\Lambda_P} \otimes S^2_{\Lambda_P}\right]\left[\mathbb{1} \otimes \mathbb{1} \otimes \hat{q}\gamma^+ S^3_F\left(M - {\Lambda_P^{-1}}^2 p_\eta\right)\right] \\ &\times \Gamma^\Lambda_M\left({\boldsymbol{\Lambda}_{\boldsymbol{P}}^{-1}}^2 \boldsymbol{p}_\xi, {\boldsymbol{\Lambda}_{\boldsymbol{P}}^{-1}}^2 \boldsymbol{p}_\eta\right).\end{aligned} \tag{87}$$

To extract the Q^2-dependence of this expression we express the boost as an exponential just like in the previous section on charge radii:

$$\begin{aligned}\langle \mathcal{P}P, \lambda' | j^+(0) | P, \lambda\rangle &= -3\int \frac{\mathrm{d}^4 p_\xi}{(2\pi)^4}\int \frac{\mathrm{d}^4 p_\eta}{(2\pi)^4} \overline{\Gamma}^\Lambda_{M,\lambda'}(\boldsymbol{p}_\xi, \boldsymbol{p}_\eta) \\ &\times \left[S^1_F\left(p_\xi + \tfrac{1}{2}p_\eta\right) \otimes S^2_F(-p_\xi + p_\eta) \otimes S^3_F(M - p_\eta)\right] \\ &\times \exp(-2\mathrm{i}\,\boldsymbol{\eta}(P)\cdot\hat{\boldsymbol{K}})\left[\mathbb{1} \otimes \mathbb{1} \otimes \hat{q}\gamma^+ S^3_F(M - p_\eta)\right] \\ &\times \Gamma^\Lambda_{M,\lambda}(p_\xi, p_\eta).\end{aligned} \tag{88}$$

This time, however, the boost generator also acts on the Dirac space of the third fermion as can be seen from the current matrix element (87):

$$\hat{K}^i = \mathrm{i}\Big[- p_\xi^0 \tfrac{\partial}{\partial p_\xi^i} - p_\xi^i \tfrac{\partial}{\partial p_\xi^0} - p_\eta^0 \tfrac{\partial}{\partial p_\eta^i} - p_\eta^i \tfrac{\partial}{\partial p_\eta^0} + \tfrac{1}{2}\left(\alpha^i \otimes \mathbb{1} \otimes \mathbb{1} + \text{cycl. perm.}\right)\Big]. \quad (89)$$

Inserting the expansion (88) into (84) and taking the limit $Q^2 \to 0$ then shows that terms with $\mathcal{O}(\boldsymbol{\eta}) > 1$ vanish because $\boldsymbol{\eta}(P)$ is of order $\sqrt{Q^2}$. Concerning the first-order term we find with the special choice of the three-momentum transfer (86) and the definition of the rapidity (56):

$$\lim_{Q^2\to 0} \frac{\boldsymbol{\eta}(P)}{\sqrt{Q^2}} = \lim_{Q^2\to 0} \frac{-\sqrt{Q^2}}{2\sqrt{M^2+Q^2/4}\sqrt{Q^2}}\boldsymbol{e}_3 = \frac{-1}{2M}\boldsymbol{e}_3. \quad (90)$$

Therefore, the static limit can safely be taken and we find for the magnetic moment:

$$\mu = -\frac{3}{4M^2}\int \frac{\mathrm{d}^4 p_\xi}{(2\pi)^4}\int \frac{\mathrm{d}^4 p_\eta}{(2\pi)^4} \overline{\Gamma}^\Lambda_{M,\lambda'}(\boldsymbol{p}_\xi,\boldsymbol{p}_\eta) \times \left[S_F^1\left(p_\xi + \tfrac{1}{2}p_\eta\right) \otimes S_F^2(-p_\xi + p_\eta) \otimes S_F^3(M - p_\eta)\right] \times \mathrm{i}\hat{K}_3\left[\mathbb{1} \otimes \mathbb{1} \otimes \hat{q}\gamma^+ S_F^3(M - p_\eta)\right]\Gamma^\Lambda_{M,\lambda}(p_\xi, p_\eta). \quad (91)$$

We inserted a factor $1/2M$ in this expression since the wave functions are normalized to $2M$ as can be seen from (19). Integration over the relative energies can now be done after replacing the fermion propagators by their partial fraction decomposition (63):

$$\langle\mu\rangle = \frac{3}{2M}\int \frac{\mathrm{d}^3 p_\xi}{(2\pi)^3}\int \frac{\mathrm{d}^3 p_\eta}{(2\pi)^3} \overline{\Gamma}^\Lambda_{M,\lambda'}(\boldsymbol{p}_\xi,\boldsymbol{p}_\eta) \times \left[\frac{\Lambda^{+++}}{(M-\Omega)} + \frac{\Lambda^{---}}{(M+\Omega)}\right]\mathrm{i}F^{3+}\hat{q}_3 \times \left[\frac{\Lambda^{+++}}{(M-\Omega)} + \frac{\Lambda^{---}}{(M+\Omega)}\right]\left[\gamma^0 \otimes \gamma^0 \otimes \gamma^0\right]\Gamma^\Lambda_{M,\lambda}(\boldsymbol{p}_\xi,\boldsymbol{p}_\eta) \quad (92)$$

with the tensor operator:

$$F^{ij} := \frac{1}{2M}\left\{\frac{p_3^j}{2\omega_3}\left[\tfrac{1}{2}(\omega_1 - \omega_2)\,\mathrm{i}\tfrac{\partial}{\partial p_\xi^i} + (\omega_1+\omega_2)\,\mathrm{i}\tfrac{\partial}{\partial p_\eta^i} - \text{h.c.}\right] + \frac{\Omega}{2\omega_3}\left(\mathbb{1} \otimes \mathbb{1} \otimes \mathrm{i}\alpha^i\alpha^j\right) + \frac{\omega_1+\omega_2}{2\omega_3^2}p_3^i p_3^j\right\}, \quad (93)$$

where we also used the anticommutator $\{\gamma^0, \alpha^i\}_+ = 0$ and the relation $\Lambda_i^\pm \alpha^j = \alpha^j \Lambda_i^\mp \pm p_i^j/\omega_i$. Note that the "+"-component in the second index of F^{ij} in (92) has to be taken in the sense of (81). Before we analyze this expression further let us replace the vertex functions in (92) by using the relations (18) and (24) to arrive at the compact notation

$$\langle\mu\rangle = \frac{3}{2M}\langle \Phi^\Lambda_{M,\lambda'}|F^{3+}\hat{q}_3|\Phi^\Lambda_{M,\lambda}\rangle. \quad (94)$$

Since F^{ij} is a product of two vector operators it constitutes a Cartesian tensor operator of rank 2, which can be decomposed into irreducible representations of the rotation group according to (67). Just as we did when deriving the charge radius, we may show that the contribution of certain representations vanish. The scalar representation gives no contribution because of the m-selection rule of the Wigner-Eckart theorem, which states that

$$\langle j_1, m_1|F_q^{[k]}|j_2, m_2\rangle = 0 \quad \text{unless} \quad m_1 - m_2 = q. \quad (95)$$

In our case $m_1 = \frac{1}{2}$, $m_2 = -\frac{1}{2}$ and $q = 0$. The spin-2 representation vanishes because of the triangularity relation

$$\langle \tfrac{1}{2}, \tfrac{1}{2}|T_q^{[2]}|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = 0. \quad (96)$$

We are thus left with the antisymmetric representation belonging to spin 1 which we may write as a vector product:

$$F^{3+[1]} = \left(F^{31} + \mathrm{i}F^{32}\right)^{[1]} = \frac{\mathrm{i}}{\sqrt{2}}\sum_{j,k=1}^{3}\epsilon_{+jk}F^{jk}, \quad (97)$$

where it is understood to take the spherical "+1"-component of the vector product as defined in (82). Note that since F^{ij} is contracted with the skew tensor ϵ_{ijk}, the last term in (93) that is proportional to $p_3^i p_3^j$ vanishes. Inserting (97) back into (94) and choosing the spin projections $\lambda' = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$ then yields:

$$\langle\mu\rangle = \frac{3}{2M}\langle \Phi^\Lambda_{M,1/2}|\tfrac{1}{\sqrt{2}}\sum_{j,k=1}^{3}\epsilon_{+jk}F^{jk}\hat{q}_3|\Phi^\Lambda_{M,-1/2}\rangle. \quad (98)$$

By using the Wigner-Eckart theorem once again we remove the spin flip and turn the expression in an expectation value:

$$\langle\mu\rangle = -\frac{3}{2M}\langle \Phi^\Lambda_{M,1/2}|\sum_{j,k=1}^{3}\epsilon_{3jk}F^{jk}\hat{q}_3|\Phi^\Lambda_{M,1/2}\rangle. \quad (99)$$

To simplify (93) further we replace the relative coordinates by one-particle coordinates:

$$F^{ij} := \frac{1}{2M}\left\{\frac{1}{2}\left[-\frac{\Omega}{\omega_3}p_3^j\left(\mathrm{i}\tfrac{\partial}{\partial p_3^i} - \frac{1}{\Omega}\sum_{\alpha=1}^{3}\omega_\alpha \mathrm{i}\tfrac{\partial}{\partial p_\alpha^i}\right) - \text{h.c.}\right] + \frac{\Omega}{2\omega_3}\left(\mathbb{1} \otimes \mathbb{1} \otimes \mathrm{i}\alpha^i\alpha^j\right)\right\}. \quad (100)$$

Since in the expectation value (99) F^{jk} is contracted with the skew symmetric tensor ϵ_{ijk} it is suggested to define:

$$\hat{L}^i_{R\alpha} := \epsilon_{ijk}p_\alpha^k\left(\mathrm{i}\tfrac{\partial}{\partial p_\alpha^j} - \hat{R}^j\right). \quad (101)$$

$\hat{L}^i_{R\alpha}$ is obviously the total angular momentum of the three-quark system with the correct center-of-mass motion removed. Furthermore, we identify the spin operator $\boldsymbol{S} = \frac{1}{2}\boldsymbol{\Sigma}$ in the following contraction:

$$\sum_{j,k=1}^{3}\epsilon_{ijk}\alpha^j\alpha^k = 2\mathrm{i}\begin{pmatrix}\sigma_i & \mathbb{1}\\ \mathbb{1} & \sigma_i\end{pmatrix} = 2\mathrm{i}\Sigma^i. \quad (102)$$

We then have

$$\sum_{j,k=1}^{3} \epsilon_{ijk} F^{jk} = -\frac{\Omega}{4M\omega_3} \left(\hat{L}^3_{Ri} + \mathbb{1} \otimes \mathbb{1} \otimes \Sigma^i + \text{h.c.} \right). \tag{103}$$

This expression is still not symmetric in the three fermions, so in the final step we symmetrize over the three fermions in the same way as we did already when deriving the charge radius:

$$\langle \mu \rangle = \frac{\langle \Phi^\Lambda_M | \hat{\mu} | \Phi^\Lambda_M \rangle}{\langle \Phi^\Lambda_M | \Phi^\Lambda_M \rangle}, \tag{104}$$

where we defined the magnetic moment operator $\hat{\mu}$ which follows from symmetrizing (103):

$$\hat{\mu} = \frac{1}{2} \left[\frac{\Omega}{M} \sum_{\alpha=1}^{3} \frac{\hat{q}_\alpha}{2\omega_\alpha} \left(\hat{L}^3_{R\alpha} + 2S^3_\alpha \right) + \text{h.c.} \right] \tag{105}$$

with the one-particle spin operators

$$\begin{aligned} \boldsymbol{S}_1 &:= \boldsymbol{\Sigma}/2 \otimes \mathbb{1} \otimes \mathbb{1}, \\ \boldsymbol{S}_2 &:= \mathbb{1} \otimes \boldsymbol{\Sigma}/2 \otimes \mathbb{1}, \\ \boldsymbol{S}_3 &:= \mathbb{1} \otimes \mathbb{1} \otimes \boldsymbol{\Sigma}/2. \end{aligned} \tag{106}$$

4.2 Interpretation

As has already been shown in the interpretation of the charge radius, the term (79) corresponds to the position of particle α as measured from the center of mass of the system. One is thus naturally led to interpret $\hat{\boldsymbol{L}}_{R\alpha}$ defined in (101) as the angular momentum (operator) observed from the relativistic center of mass. As already mentioned $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ and $\boldsymbol{S}_3$ are one-particle spin operators. We therefore conclude that the magnetic moment of the system can be decomposed in contributions of the fermion angular momenta and their spins:

$$\langle \mu \rangle = \langle \mu_L \rangle + 2\langle \mu_S \rangle, \tag{107}$$

with $\langle \mu_L \rangle$ being the contribution of the angular momenta of the three fermions:

$$\langle \mu_L \rangle := \frac{1}{\langle \Phi^\Lambda_M | \phi^\Lambda_M \rangle} \langle \Phi^\Lambda_M | \, \frac{1}{2} \left(\frac{\Omega}{M} \sum_{\alpha=1}^{3} \frac{\hat{q}_\alpha}{2\omega_\alpha} \hat{L}^3_{R\alpha} + \text{h.c.} \right) | \Phi^\Lambda_M \rangle \tag{108}$$

and $\langle \mu_S \rangle$ the contribution of the fermion spins:

$$\langle \mu_S \rangle := \frac{1}{\langle \Phi^\Lambda_M | \phi^\Lambda_M \rangle} \langle \Phi^\Lambda_M | \, \frac{\Omega}{M} \sum_{\alpha=1}^{3} S^3_\alpha \, | \Phi^\Lambda_M \rangle. \tag{109}$$

Such a decomposition into spin and angular-momentum contributions is not possible when extracting the magnetic moment from a form factor. It is thus another benefit of the approach to static properties presented in this work. In (105) we discover the same relativistic weight factor Ω/M as has already been found in the charge radius. When taking the non-relativistic limit, the operator $\hat{\mu}$ (105) becomes

$$\begin{aligned} \hat{\mu}_{\text{nr}} = \sum_{\alpha=1}^{3} \frac{\hat{q}_\alpha}{2m_\alpha} \epsilon_{3jk} p^j_\alpha \left(\mathrm{i} \tfrac{\partial}{\partial p^k_\alpha} - \frac{1}{M} \sum_{\beta=1}^{3} m_\beta \mathrm{i} \tfrac{\partial}{\partial p^i_\beta} \right) \\ + 2 \sum_{\alpha=1}^{3} \frac{\hat{q}_\alpha}{2m_\alpha} S^3_\alpha. \end{aligned} \tag{110}$$

Except for the center-of-mass correction this expression is well known. We are thus led to conclude that we have found the relativistic generalization of the non-relativistic magnetic moment operator.

Although the expression (85) we started from holds only for spin-1/2 states, one can, as in the case of the charge radius, generalize the formalism to arbitrary spins and the result is not different from the spin-1/2 case.

5 Extension to higher moments

The formalism presented so far paves the way for the calculation of higher moments as well —an issue that we briefly want to touch upon in this section. We take the electric form factor as an example and work accordingly with the "time" component of the current matrix element (26). An arbitrary moment $\langle m \rangle$ of a charge distribution is then given in general by

$$\langle m \rangle = \sum_{i_1, i_2, \ldots, i_n = 1}^{3} O_{i_1 i_2 \ldots i_n} \int \mathrm{d}^3 x \, x^{i_1} x^{i_2} \cdots x^{i_n} \rho(\boldsymbol{x}), \tag{111}$$

where $O_{i_1, i_2, \ldots, i_n}$ is a tensor of rank n, which depends on the moment to be computed. For example, for the charge radius, considered so far, O is simply

$$O_{i_1 i_2} = \frac{1}{Q} \delta_{i_1 i_2}. \tag{112}$$

By similar steps, leading from eq. (28) to eq. (43), we get

$$\begin{aligned} \langle m \rangle = \frac{1}{2M} \left(\frac{-\mathrm{i}}{2} \right)^n \sum_{i_1, i_2, \ldots, i_n = 1}^{3} O_{i_1 i_2 \ldots i_n} \\ \times \frac{\partial}{\partial P^{i_1}} \frac{\partial}{\partial P^{i_2}} \cdots \frac{\partial}{\partial P^{i_n}} \langle \mathcal{P} P | j^0(0) | P \rangle \Big|_{\boldsymbol{P}=0}. \end{aligned} \tag{113}$$

The current matrix element appearing here is defined in eq. (26). As before its P-dependent part is given by an exponential of infinitesimal boosts as in eq. (55). Because

$$\lim_{\boldsymbol{P} \to 0} \eta(P) = 0 \quad \text{and} \quad \frac{\partial}{\partial P^i} \eta^j(P) \Big|_{\boldsymbol{P}=0} = \frac{\delta_{ij}}{M}, \tag{114}$$

we find

$$\begin{aligned} \frac{\partial}{\partial P^{i_1}} \frac{\partial}{\partial P^{i_2}} \cdots \frac{\partial}{\partial P^{i_n}} \exp(-2\mathrm{i} \boldsymbol{\eta}(P) \cdot \hat{\boldsymbol{K}}) \Big|_{\boldsymbol{P}=0} \\ = \frac{(-2\mathrm{i})^n}{M^n} \hat{K}^{i_1} \hat{K}^{i_2} \cdots \hat{K}^{i_n}. \end{aligned} \tag{115}$$

Note that to every x^i from our starting equation (111) corresponds now a boost generator $\hat{K}^i$. Using this result, we get from eq. (113)

$$\begin{aligned}\langle m\rangle = &\frac{1}{2M}\sum_{i_1,i_2,\ldots,i_n=1}^{3} O_{i_1 i_2\ldots i_n}\\ &\times(-3)\int\frac{\mathrm{d}^4p_\xi}{(2\pi)^4}\int\frac{\mathrm{d}^4p_\eta}{(2\pi)^4}\overline{\Gamma}^{\Lambda}_M(\boldsymbol{p}_\xi,\boldsymbol{p}_\eta)\\ &\times\left[S^1_F\left(p_\xi+\tfrac{1}{2}p_\eta\right)\otimes S^2_F(-p_\xi+p_\eta)\otimes S^3_F(M-p_\eta)\right]\\ &\times\frac{1}{M^n}\hat{K}^{i_1}\hat{K}^{i_2}\cdots\hat{K}^{i_n}\\ &\times\left[\mathbb{1}\otimes\mathbb{1}\otimes\gamma^0\hat{q}S^3_F(M-p_\eta)\right]\Gamma^{\Lambda}_M(p_\xi,p_\eta). \qquad (116)\end{aligned}$$

Integrating out the dependence on the relative energies after replacing the propagators according to eq. (63) then results in

$$\begin{aligned}\langle m\rangle = &\frac{3}{\langle\Phi^\Lambda_M|\Phi^\Lambda_M\rangle}\\ &\times\sum_{i_1,i_2,\ldots,i_n=1}^{3} O_{i_1 i_2\ldots i_n}\langle\Phi^\Lambda_M|\frac{1}{M^n}\hat{K}'_{i_1}\hat{K}'_{i_2}\ldots\hat{K}'_{i_n}\hat{q}_3|\Phi^\Lambda_M\rangle\\ &+\text{off-diagonal matrix elements}, \qquad (117)\end{aligned}$$

where $\hat{K}'_i$ is defined in eq. (65). For $n>2$ we also find terms involving matrix elements between different energy components of the vertex function, *i.e.* between the subspaces of purely positive- and negative-energy components (denoted "off-diagonal matrix elements" in eq. (117)). Unfortunately, these terms cannot be expressed in a generic way and have to be calculated explicitly for the moment under consideration. One might, however, expect that these additional contributions are in fact small; first, because the negative-energy components correspond to the "small" components of the Dirac equation and thus vanish in the non-relativistic limit and, second, because both energy subspaces are orthogonal. Note that although the first term of eq. (117) also involves matrix elements between different energy subspaces of the Salpeter amplitudes, one can show that these do in fact vanish.

Finally, we may symmetrize the expectation value in eq. (117) over the three fermions to obtain

$$\begin{aligned}\langle m\rangle = &\frac{1}{\langle\Phi^\Lambda_M|\Phi^\Lambda_M\rangle}\sum_{i_1,i_2,\ldots,i_n=1}^{3} O_{i_1 i_2\ldots i_n}\\ &\times\langle\Phi^\Lambda_M|\sum_{\alpha=1}^{3}\hat{K}''_{i_1\,\alpha}\hat{K}''_{i_2\,\alpha}\ldots\hat{K}''_{i_n\,\alpha}\hat{q}_\alpha|\Phi^\Lambda_M\rangle\\ &+\text{off-diagonal matrix elements}, \qquad (118)\end{aligned}$$

where $\hat{K}''_{i\,\alpha}$ is defined as

$$\hat{K}''_{i\,\alpha}=\frac{1}{2}\left[\frac{\Omega}{M}\left(\mathrm{i}\frac{\partial}{\partial p^i_\alpha}-\hat{\boldsymbol{R}}\right)+\text{h.c.}\right]. \qquad (119)$$

If, *e.g.*, we insert eq. (112) into the final result (118) we instantly obtain the charge radius expression (75). What has been said about its interpretation also applies to eq. (118) in its general form.

In this sense a generalization of the formalism presented in this work to arbitrary moments is possible, although those discussed in detail, namely the charge radius and the magnetic moment, are by far the most interesting, having in addition the soundest empirical basis.

6 Application to static properties of baryons

We would like to illustrate the relevance of the preceeding theoretical considerations by applying them to an existing physical model. In refs. [1–3] a relativistic covariant quark model for baryons is treated, based on assumptions which also entered the work at hand, *i.e.* instantaneous interaction kernels and free fermion propagators corresponding to effective fermion masses. The model successfully describes mass spectra of strange and non-strange baryons up to the highest orbital and radial excitations employing a linear confinement potential and a residual interaction based on an effective instanton force. The seven parameters entering the model are fixed by a fit to the best established resonances. We use the wave functions, *i.e.* Salpeter amplitudes, that have been obtained by solving the Salpeter equation within this model to compute the expectation values of the charge radius and magnetic-moment operator derived in this work. Since no further parameters are introduced the results are genuine predictions.

6.1 Nucleon charge radii

The proton charge radius that we obtain by computation of the expectation value (75) amounts to

$$\sqrt{\langle r^2\rangle_{\text{proton}}}=0.86\,\text{fm}, \qquad (120)$$

in excellent agreement with the experimental value of $0.87\pm0.008\,\text{fm}$ from ref. [7]. The mean square charge radius of the neutron, however, results in

$$\langle r^2\rangle_{\text{neutron}}=-0.206\,\text{fm}^2 \qquad (121)$$

and overestimates the empirical number of $-0.1161\pm0.0022\,\text{fm}^2$ from ref. [7] by 77%. Within the same model the authors of ref. [4] have calculated the neutron electric form factor and extracted a mean square charge radius of $-0.11\,\text{fm}^2$ from it. The procedure, however, was numerically erroneous and a reanalysis, improving the numerical precision, resulted in a radius that is indeed compatible with our result.

6.2 Baryon octet magnetic moments

In the same model we have computed the nucleon magnetic moments using our formula (104). For the proton we find a magnetic moment of

$$\langle\mu\rangle_{\text{proton}}=2.77\,\mu_N \qquad (122)$$

Table 1. Hyperon magnetic moments compared to the empirical values.

Hyperon	Experiment [7] $[\mu/\mu_N]$	This calculation $[\mu/\mu_N]$
Λ	-0.613 ± 0.004	-0.61
Σ^+	2.458 ± 0.01	2.51
Σ^0	–	0.75
Σ^-	-1.16 ± 0.025	-1.02
Ξ^0	-1.25 ± 0.014	-1.33
Ξ^-	-0.6507 ± 0.0025	-0.56

Table 2. Prediction of magnetic moments of selected excited nucleon states. I_3 means third isospin component.

Nucleon resonance	I_3	Magnetic moment $[\mu/\mu_N]$
$P^{11}(1440)$	$1/2$	1.55
	$-1/2$	-0.98
$S^{11}(1535)$	$1/2$	0.37
	$-1/2$	-0.1

in perfect agreement with the empirical value of $2.793\,\mu_N$. The magnitude of the neutron magnetic moment

$$\langle\mu\rangle_{\text{neutron}} = -1.71\,\mu_N \tag{123}$$

is rather small if compared to the experimental magnetic moment of $-1.913\,\mu_N$.

In addition to the nucleon magnetic moments we have also calculated those of the strange octet baryons because they are experimentally well covered. Table 1 compares our results to the empirical values. The results are in excellent agreement with experiment. The largest deviation of 14% is seen with the Ξ^- magnetic moment.

Since efforts are being made to measure also magnetic moments of excited nucleon states like the $S^{11}(1535)$ as mentioned in ref. [8], we contribute some selected predictions here. The magnetic moments of the nucleon Roper resonance ($P^{11}(1440)$) and the lowest-lying state with total spin 1/2 and negative parity ($S^{11}(1535)$) are shown in table 2. The formalism allows the computation of magnetic moments of baryons with arbitrary spins and their radial excitations which will be the subject of a subsequent publication. The same is true of course for the charge radius.

As has already been indicated, the magnetic moment may be decomposed in spin and angular-momentum contributions according to eq. (107). This decomposition enables us to carry out a numerical analysis of the magnitudes of both spin and angular-momentum contributions. Note that such a study is not possible by relying on form factor calculations because there only the total magnitude of the magnetic moment can be extracted. Table 3 lists

Table 3. Contributions of quark spins ($2\langle\mu_S\rangle$) and angular momentum ($\langle\mu_L\rangle$) to the net magnetic moments of proton and neutron.

	$2\langle\mu_S\rangle$ $[\mu/\mu_N]$	$\frac{2\langle\mu_S\rangle}{\langle\mu\rangle}$ [%]	$\langle\mu_L\rangle$ $[\mu/\mu_N]$	$\frac{\langle\mu_L\rangle}{\langle\mu\rangle}$ [%]
Proton	2.53	91	0.24	9
Neutron	-1.59	93	-0.12	7

the contributions of spin and angular momentum to the magnetic moments of proton and neutron. The analysis shows, that the contribution of the quark spins exceeds the contribution of the quark angular momenta by far. There is however a small deviation from the predictions of the $SU(6)$ symmetry in the old non-relativistic quark model. The $SU(6)$ result for the ratio of proton and neutron magnetic moments emerges only at very high quark masses. One can state, however, that for constituent quark masses used in our relativistic model 90% of the magnetic moment is coming from quark spins which is due to the fact that the quarks are dominantly in a relative S-wave. This result also explains in part the success of the non-relativistic quark model in predicting the magnetic moments. Our analysis shows that by neglecting the angular motion of the quarks by assuming that the quarks are in a relative S-wave, the induced error is in the percent region. We should mention that for the $S^{11}(1535)$ the absolute value of the spin contribution is only a quarter of the angular-momentum contribution and opposite in sign. Since this resonance is dominantly a P-wave the spin has to be aligned antiparallel to the angular momentum to result in a state with total spin 1/2. The preceeding discussion is, however, only true if we work with a constituent quark mass of 330 MeV.

We may, however, carry this analysis further by studying the evolution of the spin/angular-momentum contributions with decreasing constituent quark masses. Note that the quark model described in refs. [1–3] assumes isospin symmetry between up- and down-quark and thus there is only one mass parameter for the nucleon. At different magnitudes of this mass parameter we have now fitted the remaining six parameters of the model to the baryon spectra. We might of course not expect to reproduce the spectra as well as with the original value of 330 MeV but at least we were able to keep the ground states, *i.e.* the nucleon and the Δ-particle at the empirical values. We achieved a quark mass as small as 25 MeV before numerical restrictions impeded us to go any further. Figure 2 shows the effect on the spin and angular-momentum contribution to the magnetic moment of proton and neutron. We see an almost linear decrease of the spin contribution from its original value of a good 90% at 330 MeV to roughly 60% at 25 MeV. At the same time the angular-momentum contribution gains in magnitude correspondingly to roughly 40%. Lowering the constituent quark mass in such a way should by no means be confused with an attempt to approach the chiral limit in our model. It just documents possible changes concern-

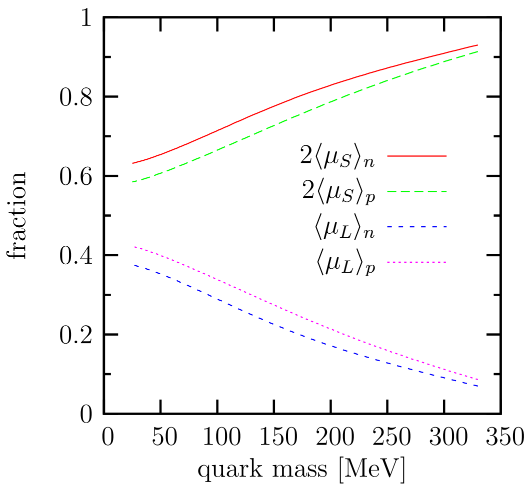

Fig. 2. Fraction of the total magnetic moment carried by quark spin and angular momentum, respectively, of proton and neutron, respectively, as a function of the quark mass.

ing magnetic-moment results when a model parameter is varied. Since our formula for the magnetic moment contains a constituent quark mass dependence which is new in comparison with non-relativistic calculations, we document here the induced changes. Nevertheless, such a variation is not completely academic. In fact, the constituent quark mass is not precisely fixed by QCD. What is fixed is the mass function and much more is now known about its momentum dependence from lattice QCD [9] since the earliest QCD computation in 1976 [10]. In the region between 0 GeV and 2 GeV it varies as a function of momentum between ~ 400 MeV and ~ 50 MeV. One may define the constituent quark mass as the value of the mass function at $p \approx 0$ but this is not compulsory; constituent quark models which aim at a description of high-mass resonances (up 3 GeV) may even be forced to use a mass value at larger momenta in order to take the full momentum dependence effectively better into account. In any case our results show that the surprisingly good $SU(6)$ predictions of the non-relativistic quark model are already accidental in a more elaborate quark picture. The true explanation of magnetic moments must in fact come from deeper results of QCD and were indeed obtained recently by chirally improved lattice calculations. Without going into details which the reader may find to be reviewed and interpreted in, *e.g.*, ref. [11] (which we quote here as a representative for the vast efforts in this field) the $SU(6)$ predictions of the non-relativistic quark model results were already shown to be accidental in the light of full QCD. These lattice results (see again ref. [11]) indicate that the values of the magnetic moments contain contributions which can be interpreted as effects of a virtual meson cloud. To describe mesonic effects in a quark model for even high-lying resonances without violating the indispensable rules of formal covariance seems to be out of reach and is perhaps even not desirable. Quark models yield an easy access to the understanding of the full hadron resonance spectrum and will at some time be replaced by lattice calculations when they become feasible.

7 Conclusion

We have shown how the charge radius and the magnetic moment of a bound three-fermion system with instantaneous interactions can be formulated as expectation values with respect to Salpeter amplitudes. The corresponding operators turned out to be natural relativistic generalizations of their non-relativistic counterparts. We also indicated how the formalism may be extended to higher moments as well. A first application of the formalism to a relativistic quark model for baryons with instantaneous interactions described in refs. [1–3] results in a good description of the nucleon charge radii and baryon octet magnetic moments except for the neutron radius. Predictions have been made for the magnetic moments of the $P^{11}(1440)$ and $S^{11}(1535)$ resonances. We found in addition an interesting dependence of the nucleon magnetic moments on the quark masses. In particular, when constituent quark masses are decreased to values almost as small as current masses, the spin contributions become equal in magnitude in comparison to the contributions of internal angular momenta. Static observables of systems with spins other than 1/2 like, *e.g.*, the baryon decuplet will be studied in the future.

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